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An Inverse Reliability-based approach for designing under uncertainty with application to robust piston design

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Abstract In this work, we propose an optimization framework for designing under uncertainty that considers both robustness and reliability issues. This approach is generic enough to be applicable to engineering design problems involving nonconvex objective and constraint functions defined in terms of random variables that follow any distribution. The problem formulation employs an Inverse Reliability Strategy that uses percentile performance to address both robustness objectives and reliability constraints. Robustness is achieved through a design objective that evaluates performance variation as a percentile difference between the right and left trails of the specified goals. Reliability requirements are formulated as Inverse Reliability constraints that are based on equivalent percentile performance levels. The general proposed approach first approximates the formulated problem via a Gaussian Kriging model. This is then used to evaluate the percentile performance characteristics of the different measures inherent in the problem formulation for various design variable settings via a Most Probable Point of Inverse Reliability search algorithm. By using these percentile evaluations in concert with the response surface methodology, a polynomial programming approximation is generated. The resulting problem formulation is finally solved to global optimality using the Reformulation–Linearization Technique (RLT) approach. We demonstrate this overall proposed approach by applying it to solve the problem of reducing piston slap, an undesirable engine noise due to the secondary motion of a piston within a cylinder.

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1 Introduction

Many engineering analyses involve optimizing a nonconvex objective function over a design space that is restricted by a set of constraints defined in terms of nonconvex functions. An application of standard nonlinear optimization methods to such a problem can at best attain a local solution that need not be a global optimum. In addition, the application of deterministic approaches to design does not consider the impact of uncertainties. The resulting design solution may either be too sensitive to system variations causing a loss of system performance, or be unreliable and violate critical design constraints. In contrast, robust design and probabilistic or reliability-based design represent two major paradigms for designing under uncertainty. Robust design focuses on improving the quality of the product by minimizing the effect of variation without eliminating the causes. On the other hand, probabilistic or reliability-based design focuses on maintaining design feasibility with respect to system performance measures at specified probabilistic levels.

More specifically, a typical problem of designing under uncertainty within the paradigm of probabilistic or reliability based design optimization can be stated as follows:

$$\begin{array}{ll} \text{Minimize} & v_{G_{obj}} \equiv f\,(x,\,y,\,P) \\ \text{subject to} & \text{Prob}[G_i\,(x,\,y,\,P) \leq 0] \geq \alpha_i, \quad \forall i = 1,\ldots,m \\ & (\nu_x, y) \in Z \end{array} \tag{1}$$

where, x is the vector of *random design variables*, or *control factors*, as governed by the vector of distribution parameters, v_x, which might include the mean, the standard deviation, etc.; y is the vector of *deterministic design variables*; P is the vector of *random design parameters* or *noise factors*, which is governed by the vector ν_P of its specified distribution parameters; G_{obj} is the actual underlying objective function (as a function of x, y, and P), which is generally a random variable; $v_{G_{\text{obj}}}$ is a probabilistic characteristic of the objective function, e.g., its mean or standard deviation, or some combination of these measures; in robust design, robustness is usually achieved via an objective that simultaneously attempts to optimize the mean performance and minimize the performance variation. f is the design objective, which describes the probabilistic characteristic $v_{G_{\text{obj}}}$ of the objective function G_{obj}; Prob[·] denotes probability of [·]; $G_i, \forall i$ are probabilistic constraint functions of x, y, and P; $\alpha_i, \forall i$ are the desired probabilities of constraint satisfaction; Z is a constraining set that controls the feasible values of ν_x and y, typically containing lower-upper bounding (and sometimes integrality) restrictions.

Here, $\{v_x, y\}$ are jointly considered as *design variables*, and represent the principal decision variables that control the final design outcome. Also, note that if $G_i^{\alpha}\,(x, y, P)$ denotes the α-percentile level of the probabilistic function $G_i\,(x, y, P)$, i.e., $\text{Prob}[\,G_i\,(x,\,y,\,P) \leq G_i^{\alpha}\,(x, y, P)] = \alpha$ as governed by some underlying probability density function of $G_i\,(x, y, P)$, then conceptually, we can equivalently state (1) as the restriction

$$G_i^{\alpha_i}\,(x,\,y,\,P) \,\leq\, 0, \quad \forall i = 1,\ldots,m. \tag{2}$$

In this work, we adopt an approach that integrates robustness and reliability considerations within a model of the type (1) via the Inverse Reliability Strategy (see Du et al. 2004), and we investigate the composition of various modeling, approximation, and global optimization techniques towards developing an optimization framework for designing under uncertainty. We then apply this approach to the problem of minimizing piston slap, an undesirable engine noise due to the secondary motion of a piston within a cylinder.

The motivation and specific contributions of this work are threefold. The first is to provide a pedagogical description of the Inverse Reliability strategy for modeling probabilistic design optimization problems of the type (1), along with a discussion of various analytical tools that can be utilized to characterize the objective and constraint functions. Second, we propose a new overall framework for designing under uncertainty that composes these diverse tools in order to transform the probabilistic design problem into a nonconvex mathematical program, and then coordinates this with a suitable global optimization technique. Third, as an illustration of this approach, we apply it to solve the aforementioned piston slap problem, which was posed to us by Ford Motor Company. Overall, we hope that this exposition will stimulate the global optimization community to explore alternative approaches to solve such probabilistic design problems to (near-) global optimality, a feature that has been lacking in the literature and deserves more attention than has been forthcoming.

The remainder of this paper is organized as follows. Section 2 provides an overview of the Inverse Reliability strategy, and Section 3 outlines the Most Probable Point concept for evaluating the percentile performance characteristics for the objective and constraint functions. Section 4 briefly discusses certain relevant approximation methods commonly employed in engineering design. The foregoing sections therefore provide a concise tutorial on the basic concepts that constitute the elements of our proposed overall approach. As an illustrative application analyzed herein, Section 5 describes the robust piston design problem or the piston slap problem in automobile engine design. Section 6 then delineates the proposed optimization framework for designing under uncertainty, and presents computational results pertaining to applying this approach to solve the piston slap problem. Section 7 concludes the paper with some comments and recommendations for future research.

2 The Inverse Reliability Strategy for design optimization under uncertainty

The Inverse Reliability Strategy uses percentile performance to assess both robustness objectives and reliability constraints. Robustness is achieved through a design objective that evaluates performance variation as a percentile difference between the right and left trails of the specified goals. Reliability requirements are formulated as inverse reliability constraints that are predicated on equivalent percentile levels. This reformulation of the traditional probabilistic optimization formulation is motivated by the need to facilitate more computationally efficient techniques, and to provide a more accurate assessment of performance dispersion from the viewpoint of improving the system robustness. The resulting formulation represents a multicriteria optimization problem where a trade-off needs to be made between optimizing the mean performance and minimizing the performance variation, subject to probabilistic chance-constraints. Usually, the multiple objectives are combined into a single objective function by using some suitable normalized weighting factors, as we shall later illustrate in Section 5 when addressing the piston slap problem.

In conventional reliability analysis, the focus is on determining if the probability of some system performance measure is greater than, or less than, a specified level. In Inverse Reliability or percentile formulations, the focus is on designing a system that yields such a specified reliability performance. Du et al. (2004) provide a detailed derivation of the Inverse Reliability formulation. Such a percentile formulation has several advantages:

(i) It represents probability at the tail-areas of the system performance distribution and, hence, carries more information than the standard deviation. For example, the percentile performance captures the skewness of the distribution, whereas the standard deviation captures only the dispersion around the mean value.
(ii) It naturally provides the confidence level with which the design robustness is achieved.
(iii) Computationally efficient solution procedures can be derived based on using such percentile representations for both the robustness objective and the probabilistic constraints.

The Inverse Reliability formulation can be stated as the following multiple objective program, in its generic form:

$$\begin{aligned} \text{IR Minimize } & \left\{ \mu_{G_{\text{obj}}}(x, y, P),\ \Delta G_{\text{obj}}(x, y, P)_{\alpha_1}^{\alpha_2} \right\} \\ \text{subject to } & G_i^{\alpha_i}(x, y, P) \leq 0, \quad \forall i = 1, \ldots, m \\ & (v_x, y) \in Z \end{aligned} \tag{3}$$

where $\mu_{G_{\text{obj}}}(x, y, P)$ represents the mean of $G_{\text{obj}}(x, y, P)$, and where $\Delta G_{\text{obj}}(x, y, P)_{\alpha_1}^{\alpha_2} \equiv G_{\text{obj}}^{\alpha_2}(x, y, P) - G_{\text{obj}}^{\alpha_1}(x, y, P)$ represents the difference in the percentile values at the probability levels α_2 and α_1. Here again, $\{v_x, y\}$ are the design variables, where the distribution parameter vector v_x controls the random design variable vector x, and all other notation is as described above.

As an alternative to the objective function given in (3), for a smaller-the-better type of robust design, the objective function could be formulated as: Minimize $G_{\text{obj}}^{\alpha}(x, y, P)$, where α is a large probability, for example, 0.95 or 0.99. The percentile value at the right-tail of the distribution of $G_{\text{obj}}(x, y, P)$ is to be minimized in this case. Symmetrically, for a larger-the-better type of robust design, the design objective could be formulated as: Maximize $G_{\text{obj}}^{\alpha}(x, y, P)$, where α is a small probability, for example, 0.05 or 0.01, and where the intent is to maximize the percentile value at the left-tail of the distribution of $G_{\text{obj}}(x, y, P)$.

The key requirement for being able to solve Problem IR defined in (3) is to have an analytical facility to evaluate the defining percentile values in the objective and constraint functions, for any specified probability level, given the distribution characteristics (v_x, y, v_p), and assuming some underlying probability distribution. This is addressed in the following section.

3 Most probable point of inverse reliability

One of the challenges in using probabilistic design models is to capture the effect of uncertainty on a system response, given the probability distributions of the random input variables. Several approaches have been adopted for such uncertainty analyses. Some of the more commonly employed methods are outlined here.

Sensitivity-based approximation procedures include the worst-case analysis and the moment matching method (see Eggert 1991; Parkinson et al. 1993; Du and Chen 2000, 2002). In the worst-case analysis, all fluctuations are assumed to occur in the worst possible combination and the Taylor series expansion (or an optimization subproblem) is used to compute the worst value of the system output. In the moment matching method, the first-order moment (mean value) and the second-order moment (standard deviation) of the system output are obtained and are used to determine the resulting probability distribution. The moment matching method is not accurate enough for large input uncertainties. Also, low-order moments fail to accurately capture the resulting probability distributions.

Monte Carlo simulation can also be used to generate the cumulative distribution function and the probability density function of a system response based on data sampling. However, this can be computationally prohibitive. Although some methods have been proposed to improve the computational efficiency of Monte Carlo simulation, such as Latin Hypercube sampling (Walker 1996), the shooting Monte Carlo approach (Brown and Sepulveda 1997), and the directional simulation technique (Ditlevsen et al. 1987), the required computational effort can still be excessive. Models constructed via the Response Surface Methodology (RSM) somewhat alleviate the computational expense involved in Monte Carlo simulation, but they tend to overly smooth the response behavior, and miss local sensitivities and relatively finer variations in the probabilistic analysis.

Reliability analysis-based approaches have emerged as a promising alternative for uncertainty analysis. They are generally more accurate than sensitivity-based approximations and RSM models, and are more efficient than sampling-based methods. Reliability analysis methods employ analytical techniques to find a particular point in the design space that can be related (at least approximately) to the probability of a system response being less than some limit state value (see Melchers 1999). This point is often referred to as the *Most Probable Point*. Note that in order to be able to solve the Problem IR formulated in (3), we need to be able to determine the percentile value, $g^{\alpha}(x, y, P)$, say, for any given objective or constraint function $g(x, y, P)$, and any given distribution characteristics (v_x, y, v_P), for any specified probability level α. The Most Probable Point approach provides the facility to compute this value.

To describe this methodology, consider any objective or constraint function, written generically in terms of random design variables $x = (x_1, x_2, \ldots, x_n)$, as $g(x)$. (In Section 4, we describe a Kriging approach for deriving the form of $g(x)$, given appropriate sample data. Also, note that we are focusing here on the random design variables x for simplicity in exposition; the random design parameter P can be treated similarly, and the deterministic design variables could be assumed to be fixed at some suitable specific values.) Suppose that based on a selected design variable characteristic v_x, each x_i has some known resultant distribution with a cumulative distribution function given by $F_i\,(x_i)$, $\forall\, i = 1, \ldots, n$. Now, given $g(x)$ with this known distribution for x (based on a specified v_x), and given a probability level α, we need to find the percentile value

$$c \equiv g^{\alpha}, \quad \text{where } \operatorname{Prob}[g(x) \leq c] = \alpha. \tag{4}$$

The Most Probable Point method accomplishes this by estimating the cumulative distribution of g as follows. First, it adopts a transformation from the x-space to the

u-space of standard normal variates via the relationships

$$u_i = \Phi^{-1}[F_i(x_i)], \quad \forall i = 1, \ldots, n, \tag{5}$$

where Φ represents the cumulative distribution function of a standard normal distribution. This transforms $g(x)$ to a function of u, say $\hat{g}(u)$. Then, (4) reduces to the problem of finding c such that

$$\text{Prob}[\hat{g}(u) \le c] = \alpha. \tag{6}$$

Note that in case $\hat{g}(u)$ is linear (see Fig. 1), then, given a value of c, if we compute

$$\beta = \min\left\{\|u\| : \hat{g}(u) = c\right\}, \tag{7}$$

we would have (see Du and Chen 2001)

$$\alpha = \text{Prob}[\hat{g}(u) \le c] = \begin{Bmatrix} \Phi(\beta) & \text{if } \alpha \ge 0.5 \\ 1 - \Phi(\beta) & \text{if } \alpha < 0.5 \end{Bmatrix}. \tag{8}$$

The Most Probable Point method assumes that (8) is approximately true even when $\hat{g}$ is nonlinear. Moreover, note that we have the inverse situation at hand of finding c, given α. Hence, based on (8), we first compute

$$\beta = \begin{Bmatrix} \Phi^{-1}(\alpha) & \text{if } \alpha \ge 0.5 \\ \Phi^{-1}(1-\alpha) & \text{if } \alpha < 0.5 \end{Bmatrix}. \tag{9}$$

Next, we find a Most Probable Point, given by u_{MPP}, which solves the following problem (based on (7) and (8)):

$$u_{\text{MPP}} \text{ solves} : \begin{Bmatrix} \max\{\hat{g}(u) : \ \|u\| = \beta\}, & \text{if } \alpha \ge 0.5 \\ \min\{\hat{g}(u) : \ \|u\| = \beta\}, & \text{if } \alpha < 0.5 \end{Bmatrix}. \tag{10}$$

Finally, we compute

$$c = g^{\alpha} = \hat{g}(u_{\text{MPP}}). \tag{11}$$

Several optimization algorithms such as Sequential Quadratic Programming, Sequential Linear Programming, and the Modified Method of Feasible Directions (see Bazaraa et al. 1993); or specialized search procedures as described in Ditlevsen

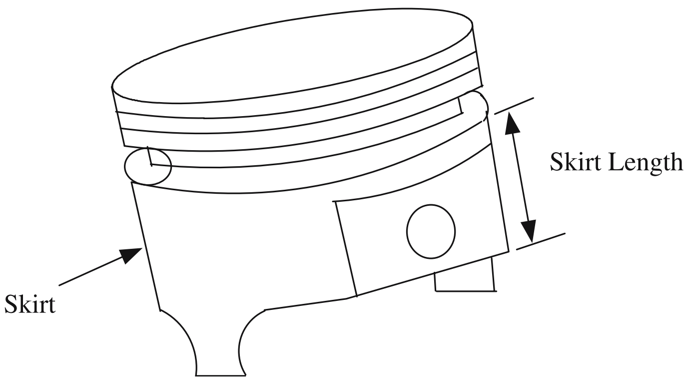

Fig. 1 Transformation of input variables and percentile computation

and Madsen (1996), Du and Chen (2001), and Yuon et al. (2003) can be used to solve (10). However, these algorithms may not guarantee an optimal solution for nonconvex functions or might experience numerical convergence problems in general. We recommend using the Most Probable Point of Inverse Reliability search algorithm of Du et al. (2004), which has been shown to achieve robust convergence for nonconvex problems. This algorithm detects the Most Probable Point as the common point lying on the tangent to the β-sphere and the contour of the function $\hat{g}(u)$ in the u-space by specifically exploiting the structure of this problem. At this point, the vector connecting it to the origin is in the direction of the negative gradient of $\hat{g}(u)$(for the maximization case in (10); the minimization case can be considered similarly). To achieve this optimality condition, the search first adopts the steepest ascent direction. When this direction does not lead to an increased function value, a secondary search process along the arc of the β-sphere is implemented. This latter step, which specifically exploits the special structure of (10), imparts this procedure its relatively superior convergence behavior, particularly for nonconvex functions. Therefore, this search algorithm turns out to be robust for various types of functions.

We next describe techniques for deriving the form of $g(u)$, given a set of responses at a collection of sample data points.

4 Approximation methods used in the simulation–optimization framework

Engineering analyses often involve modeling system performance via techniques such as finite element methods and computational fluid dynamics, both of which entail heavy computational requirements. As a result, high fidelity analyses can become computationally prohibitive, thereby limiting optimization and design space explorations. Consequently, statistical approximation procedures are becoming increasingly popular for constructing simplified surrogate approximations or metamodels of these analytical codes. Simpson et al. (2002) identify a variety of methods that have been developed to model and assess the effects of uncertainties by converting deterministic problem formulations into probabilistic formulations.

The Response Surface Methodology (RSM) (see Myers 1995) is a very popular choice for constructing metamodels, especially in the aerospace industry and in structural design. This method typically employs second-order polynomial models that are fit using least-square regression techniques. While this is analytically convenient, its accuracy is usually acceptable only over a relatively small region of the variable space, particularly for complex functions or multiple objective approximations (see Barton 1992; Koch et al. 1999).

On the other hand, Kriging models show great promise for building accurate global approximations over potentially large regions of interest. They are extremely flexible because of the wide range of spatial correlation functions that can be chosen to build the approximations, provided that sufficient sample data are available to capture the trends in the system response. Furthermore, Kriging models can either honor the data by providing an exact interpolation of the data, or smooth the data by providing an inexact interpolation (see Cressie 1993; Montes 1994). Booker (1998) discusses a 56 variable helicopter-rotor structural design problem, and demonstrates how the flexibility of Kriging models permits such representations to be improved iteratively in regions of interest through an intelligent intervention of a design expert. Osio and Amon (1996) also present a multi-stage Kriging strategy to design an embedded electronic package involving five design variables.

The limited use of Kriging models in engineering applications may be attributed to the lack of readily available software to fit Kriging models, or the additional effort involved in using a Kriging model as compared to a simple RSM model. Kriging models combine a global model plus localized departures via the functional form:

$$g(x) = h(x) + \xi(x) \tag{12}$$

where,

- $g(x)$ is the unknown function of interest;
- $h(x)$ is an approximation (usually polynomial) function, and
- $\xi(x)$ is the realization of a stochastic process with mean zero, variance σ^2, and nonzero covariance.

The term $h(x)$ provides a base model for the design space, and is similar to a polynomial response surface. In many cases, however, $h(x)$ is taken as simply a constant value. $\xi(x)$ creates localized deviations so that the Kriging model interpolates the n_s sample data points; however, noninterpolative Kriging models can also be created to smooth noisy data. The covariance matrix of $\xi(x)$ is given by

$$\mathrm{Cov}\,[\xi(x^i), \xi(x^j)] = \sigma^2[R(x^i, x^j)] \tag{13}$$

where, $R(x^i, x^j)$ is the correlation function between any two of the n_s sample data points x^i and x^j. Let R be the correlation matrix having components $R(x^i, x^j)$. Note that R is an $(n_s \times n_s)$ symmetric matrix with ones along the diagonal. The correlation function $R(x^i, x^j)$ is specified by the user and a variety of such functions exist. Often-times, a Gaussian correlation function of the following form is used.

$$R(x^i, x^j) = \exp\left[-\sum_{k=1}^{n} \theta_k \left|x_k^i - x_k^j\right|^2\right] \tag{14}$$

where,

- n is the number of design variables;
- $\theta_k, k = 1, \ldots, n$, are the unknown correlation parameters used to fit the model, and
- x_k^i and x_k^j are the kth components of the sample points x^i and x^j, respectively.

In some cases, using a single correlation parameter gives sufficiently good results; however, in our approach, we use a different θ_k for each design variable $x_k, k = 1, \ldots, n$. The predicted estimates, $\tilde{g}(x)$, of the response $g(x)$ at untried values of x are then given by (see Cressie 1993 for details)

$$\tilde{g}(x) = \tilde{\gamma} + r^T(x)\, R^{-1}\, (g - h\tilde{\gamma}) \tag{15}$$

where,

- g is the column vector of length n_s that contains the sample values of the response;
- h is a column vector of length n_s that is filled with ones (this assumes that $h(x)$ is taken as a constant);
- $r^T(x)$ is the transpose of the correlation vector, of length n_s between an untried x and the sampled data points $\{x^1, \ldots, x^{n_s}\}$ given by

$$r^T(x) = [R(x, x^1), R(x, x^2), \ldots, R(x, x^{n_s})]; \tag{16}$$

- $\tilde{\gamma}$ is estimated as

$$\tilde{\gamma} = \left(h^T R^{-1} h\right)^{-1} h^T R^{-1} g. \tag{17}$$

The estimate of the variance, $\tilde{\sigma}^2$, between $\tilde{\gamma}h$ and g is then computed using the following equation:

$$\tilde{\sigma}^2 = [(g - h\,\tilde{\gamma})^T R^{-1}(g - h\,\tilde{\gamma})]/n_s. \tag{18}$$

The maximum likelihood estimates for the θ_k-parameters in (14) that are used to fit this Kriging model are obtained by solving the following problem:

$$\text{Maximize } \{-[n_s \ln(\tilde{\sigma}^2) + \ln\|R\|]/2 : \quad \theta \geq 0\}, \tag{19}$$

where $\theta = (\theta_1, \ldots, \theta_n)$, and where both $\tilde{\sigma}^2$and $\|R\|$ (the norm of the matrix R), are functions of θ. While any value of θ creates an interpolative Kriging model, the ideal Kriging model is found by solving the nonlinear optimization problem given by (19). In our approach, we employed a simulated annealing-based algorithm to find the maximum likelihood estimates for the correlation parameters θ via (19). The surrogate model used for g was then given by (15).

5 Illustrative application: The design of combustion engines to reduce piston slap under uncertainty

In automobile vehicle design, total vehicle customer satisfaction is strongly linked to the level of satisfaction a customer has with the vehicle's engine. One of the key elements of vehicle satisfaction is the Noise, Vibration, and Harshness characteristics of the vehicle and the engine. Piston slap is an unwanted engine noise that is the result of the secondary motion of a piston, i.e., the departure of the piston from the nominal motion prescribed by the slider crank mechanism. This secondary motion is caused by a combination of transient forces and moments acting on the piston during engine operation and the presence of clearances between the piston and the cylinder liner. This combination results in both a lateral movement of the piston within the cylinder and a rotation of the piston about the piston pin, and it causes the piston to impact the cylinder wall at regular intervals. These impacts result in an objectionable engine noise known as piston slap.

Our formulation of the engine design problem described below is based on physical test data obtained from Ford Motor Company (this data pertains to some 21 operating points measured at 1200 RPM). There are four random design variables (control factors) and two random design parameters (noise factors) in this design. The four random design variables are designated as follows.

1. *Skirt Length* (x_1): The piston skirt is the area along the circumference of the piston, below the piston rings, which carries the inertial side loads from the piston. Figure 2 illustrates this region and depicts the segment that defines the skirt length.
2. *Skirt Profile* (x_2): There is an inherent asymmetry in the piston assembly that gives rise to a non-flat surface as assessed vertically between the top and bottom of the skirt. The control of this profile is managed by specifying an associated dimensionless integer variable that takes on values 1, 2, and 3, depending on the particular skirt profile available through the manufacturing process.

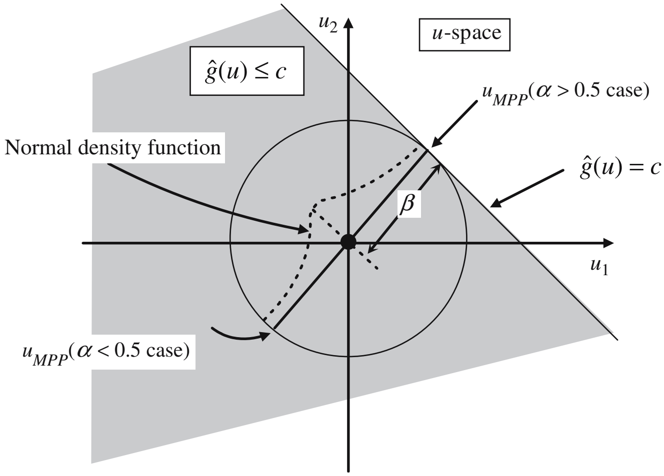

Fig. 2 Schematic illustration of piston

3. *Skirt Ovality* (x_3): The asymmetric piston assembly leads to non-uniform temperature distortions and asymmetrical expansions. To ensure uniform, minimal clearances under all operating conditions, the piston is accurately machined to a non-circular or oval shape. This machining is also controlled by the specification of a dimensionless integer variable, which takes on values 1, 2, and 3, depending on the particular shape available through the manufacturing process.
4. *Pin Offset* (x_4): In a piston assembly, the center of the piston pin is always offset from the center of the piston. This offset is provided since it somewhat reduces the effect of the inertial side loads carried by the piston skirt from the forces acting on the piston.

The two noise factors (design parameters) are delineated as follows.

1. *Clearances between the Piston and the Cylinder Liner* (P_1): The thermal coefficient of expansion of the piston is always greater than that of the cylinder bore. Sufficient clearance between the piston and the cylinder liner is necessary to prevent the piston from seizing when operating at its maximum possible service temperature.
2. *Location of Peak Pressure* (P_2): The location of peak pressure refers to the position of the piston within the cylinder, measured in degrees, after it crosses the top dead center.

Note that no deterministic design variables exist in this problem. The random design variables or control factors in the problem are governed by their mean values, (or prescribed index values in the case of the skirt profile and skirt ovality variables). These are designated as follows: v_1, mean skirt length (mm); v_2, skirt profile index (integer value, dimensionless); v_3, skirt ovality index (integer value, dimensionless), and v_4, mean piston pin offset (mm).

Additionally, the random design parameters in the problem are specified via their mean values, which are designated as follows: v_5, mean clearance between the piston and the cylinder liner (μm), and v_6, mean peak pressure location (degrees after crossing the top dead center).

The robust design objective is to minimize the noise (G_{obj}) and its variation. The friction (G_1) is considered via a reliability constraint, which requires that the probability of the friction G_1 being lesser than 7 N should be at least 0.99. Using the proposed Inverse Reliability Strategy (percentile formulation for both objective and constraints), we can represent the piston slap problem (PSP) in the following generic form:

$$\text{(PSP)} \quad \text{Minimize} \quad w_1 \frac{\mu_{G_{\text{obj}}}}{\mu^*_{G_{\text{obj}}}} + w_2 \frac{\left(\Delta G_{\text{obj}}\right)_{0.05}^{0.95}}{\left(\Delta G^*_{\text{obj}}\right)_{0.05}^{0.95}} \tag{20a}$$

$$\text{subject to} \quad G_1^{0.99} - 7 \leq 0 \tag{20b}$$

$$l_j \leq v_j \leq u_j, \quad \forall j = 1, \ldots, 4, \text{ with } v_2 \text{ and } v_3 \text{ integer-valued} \tag{20c}$$

$$v_j \text{ pre-selected in } [l_j, u_j], \text{ for } j = 5, 6, \tag{20d}$$

where,

- $\mu_{G_{\text{obj}}}$ is the mean value of the piston noise;
- $\left(\Delta G_{\text{obj}}\right)_{0.05}^{0.95}$ is the percentile variation of the piston noise;
- v_j are the distribution characteristics for the random design variables for $j = 1, \ldots, 4$, and for the random design parameters for $j = 5, 6$;
- (l_j, u_j), $j = 1, \ldots, 6$, are specified bounds on the respective distribution characteristics for the design variables and parameters $v_j, \forall j = 1, \ldots, 6$, as displayed in Table 1;
- w_1 and w_2 are relative weighting factors for the two objective terms, and
- $\mu^*_{G_{\text{obj}}}$ and $\left(\Delta G^*_{\text{obj}}\right)_{0.05}^{0.95}$ are ideal solution values that are used to normalize the two respective terms in the objective function, and are respectively obtained by separately minimizing $\mu_{G_{\text{obj}}}$ and $\left(\Delta G_{\text{obj}}\right)_{0.05}^{0.95}$ subject to (20 b, c, d).

6 Proposed optimization framework and its application to solve the piston slap problem

The proposed overall approach for solving the general Inverse Reliability formulation (3) (where some composite function for the multiple objectives is employed) adopts the following steps:

Step 1. Derive surrogate functional forms for the performance measures required to model Problem IR (whenever not available) by applying the Gaussian Kriging technique described in Section 4 using an available set of sample

Table 1 Bounds on the distribution characteristics for the Random Design Variables and Design Parameters.

Variable v_j	Lower Bound l_j	Upper Bound u_j
v_1 (continuous)	21	25
v_2 (discrete)	1	3
v_3 (discrete)	1	3
v_4 (continuous)	0.5	1.3
v_5 (continuous)	15	85
v_6 (continuous)	12	18

data points and corresponding response evaluations. (Note that these data points pertain to particular realizations of (x, y, P) based on physical tests conducted on available specimen designs.)

Step 2. Evaluate the percentile performance characteristics in the problem formulation via the Most Probable Point of Inverse Reliability (see Du et al. (2004)) search algorithm as discussed in Section 3, for various design variable distribution character settings (i.e., values of (v_x, y, v_P)), chosen via an appropriate experimental design. A Fractional Factorial Design (see Hicks 1973) can be used for this purpose.

Step 3. Using the percentile evaluations in Step 2, apply the Response Surface Methodology (see Myers 1995) to generate a polynomial programming approximation for each percentile-based term in the objective and constraint functions of Problem IR. Let the resulting polynomial programming nonconvex optimization problem, of minimizing a polynomial objective function subject to polynomial constraints based on the foregoing approximations, be designated as Problem PPA.

Step 4. Solve Problem PPA to global optimality using, for example, the Reformulation–Linearization Technique (RLT) of Sherali and Tuncbilek (1992), or the software package BARON developed by Sahinidis (1996). Alternatively, for more complex problems, one of the following two heuristic approaches can be adopted at this step.

(a) *Heuristic MINLP*: Apply a local search-based (mixed-integer) nonlinear programming (MINLP) procedure (see Kocis and Grossmann 1989 or Viswanathan and Grossmann (1990), for example) to Problem PPA.

(b) *Heuristic RLT-LP-MINLP*: Derive the RLT-based, higher dimensional, linear programming relaxation RLT-LP of Problem PPA (see Sherali and Tuncbilek 1992). Solve RLT-LP, and then apply a MINLP algorithm to refine the resulting solution.

We now apply the foregoing procedure to the piston slap problem PSP described in Section 5 (see Equation (20)). A flow-chart for the proposed algorithmic framework outlined in steps 1 through 4 above, as translated for this problem, is depicted in Fig. 3. For the purpose of illustration, the mean values for the random design parameters P_1 and P_2 were selected as $v_5 = 50$ and $v_6 = 17$, respectively. Applying Steps 1–3, the particular forms of the polynomial approximations for the mean and for the percentile difference related to the engine noise, which constitute the objective function, and for the mean of the engine friction measure, which constitutes the constraint function, were obtained as specified below by (21), (22), and (23), respectively.

$$\begin{aligned}\mu_{G_{\text{obj}}} = {} & 2.4635v_1 + 1.6963v_2 - 6.7356v_3 + 6.7356v_4 - 0.0524v_1^2 - 0.1684v_1v_2 \\ & + 0.0408v_1v_3 - 0.3907v_1v_4 + 0.5463v_2^2 - 0.0019v_2v_3 - 0.1522v_2v_4 \\ & + 0.7707v_3^2 + 1.7499v_3v_4 - 1.2979v_4^2 + 38.3897. \end{aligned} \tag{21}$$

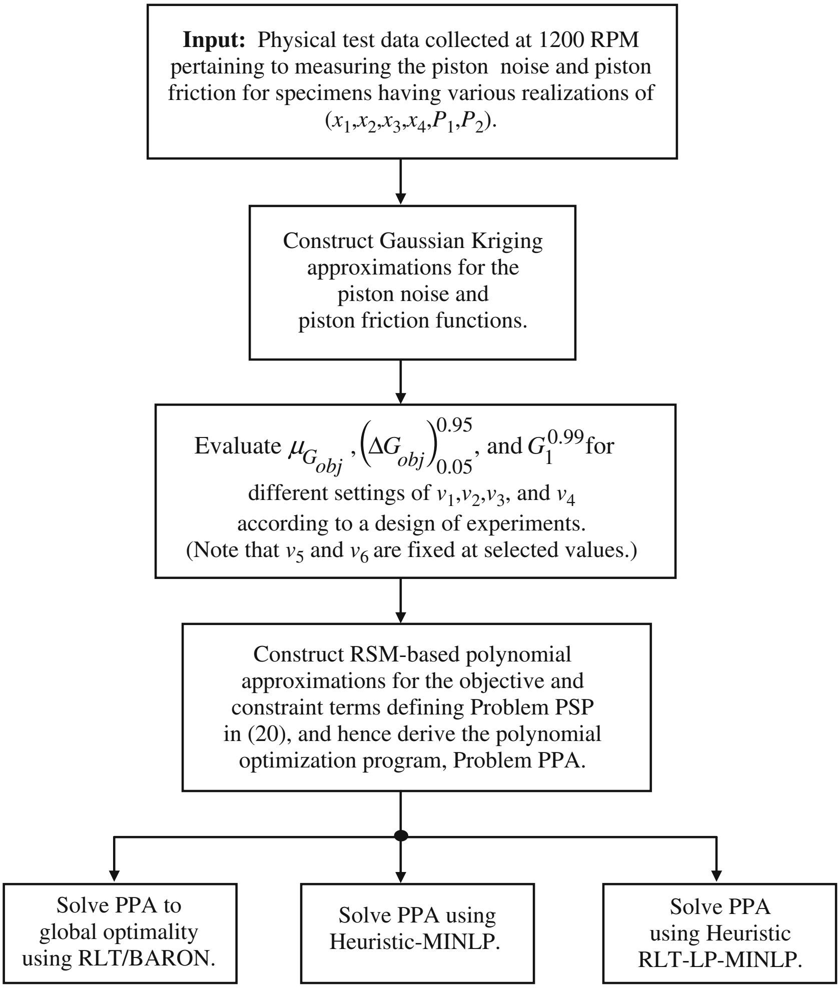

Fig. 3 Flow-chart for the Proposed Algorithm applied to Problem PSP

$$
\begin{aligned}
\left(\Delta G_{\text{obj}}\right)_{0.05}^{0.95} = &-2.9556v_1 - 1.8282v_2 - 5.5577v_3 + 5.1907v_4 + 0.0668v_1^2 \\
&+ 0.0918v_1v_2 + 0.2072v_1v_3 - 0.2783v_1v_4 + 0.0161v_2^2 \\
&+ 0.1326v_2v_3 - 0.2375v_2v_4 + 0.1318v_3^2 - 0.1169v_3v_4 \\
&- 0.0716v_4^2 + 37.1626.
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
G_1^{0.99} = &\ 0.3324v_1 + 6.9473v_2 - 11.2527v_3 + 4.6857v_4 - 0.000928v_1^2 \\
&- 0.3195v_1v_2 + 0.4279v_1v_3 - 0.5993v_1v_4 + 0.0791v_2^2 \\
&- 0.2514v_2v_3 + 0.8699v_2v_4 + 0.4810v_3^2 + 0.1797v_3v_4 \\
&+ 3.3915v_4^2 + 6.6784.
\end{aligned} \tag{23}
$$

The goodness-of-fit statistics in terms of the R^2-values (see Hines and Montgomery 1972), for these functions are displayed in Table 2, and reveal a relatively high level of fidelity.

Table 3 outlines the results obtained by using RLT to solve the resulting polynomial program PPA, which is given by (20), using the approximations (21)–(23). Note that the first two rows of Table 3 relate to respectively minimizing $\mu^*_{G_{\text{obj}}}$ and $(\Delta G^*_{\text{obj}})^{0.95}_{0.05}$ for defining the objective function (20a) for Problem PSP. The last row then provides the results for this latter problem. The columns for RLT-LP pertain to solving the initial LP relaxation generated by RLT using CPLEX Version 9.0, and those for RLT-PPA refer to solving PPA to optimality using RLT (identical results were obtained using BARON). We also tried solving PPA using Heuristic MINLP as delineated in Step 4(a) by applying DICOPT (see Viswanathan and Grossman 1990; Kocis and Grossmann 1989) in concert with the General Algebraic Modeling System (GAMS), Version 21.3, developed by the GAMS Development Corporation (see http://www.gams.com), using the starting solution specified by $(v_1, v_2, v_3, v_4) = (25, 2, 2, 1)$. In addition, we applied the Heuristic RLT-LP-MINLP as described in Step 4 (b). Both these runs produced the same solution as that given in Table 3 for RLT-PPA, indicating that the heuristic local search methods effectively handled this problem instance. The following are the performance characteristics achieved at the optimal solution: Noise, Mean = 56.5913; Noise, Percentile Difference = 2.3858; Friction, 99 Percentile = 6.99998407.

The mean piston noise and friction values identified by this algorithmic framework are lesser than the current design used at Ford Motor Company, indicating the practical usefulness of using this approach. Actual manufacturing implications and implementation issues for applying this solution to the piston design process require further considerations.

Table 2 Goodness-of-Fit Statistics for the Polynomial Programming Approximations

Term	R^2 Value
$\mu_{G_{\text{obj}}}$	0.999614
$\left(\Delta G_{\text{obj}}\right)^{0.95}_{0.05}$	0.979304
$G_1^{0.99}$	0.999515

Table 3 Solution for PPA using the RLT algorithm

Solution Step	RLT-LP				RLT-PPA			
	(ν^*) Values	Objective Value	Iters.	Time (sec)	(ν^*) Values	Objective Value	Iters.	Time (sec)
$\mu^*_{G_{\text{obj}}}$	{25, 2, 2, 1.3}	52.1053	20	0.001	{25, 2, 2, 1}	54.5567	1	0.001
$\left(\Delta G^*_{\text{obj}}\right)^{0.95}_{0.05}$	{23, 2, 2, 1.3}	0.8755	15	0.002	{24.5628, 1, 1, 1.3}	2.3859	1	0.002
Problem PSP	{23, 2, 2, 1.3}	1.0222	15	0.001	{24.5628, 1, 1, 1.3}	1.0129	1	0.001

7 Conclusions and recommendations for future research

In this paper, we have presented an optimization framework for designing under uncertainty employing the Inverse Reliability Strategy that uses percentile performance to assess both robustness objectives and reliability constraints. Our overall approach prescribes a novel composition of several analytical tools: approximation techniques based on Gaussian Kriging models, percentile evaluations using the Most Probable Point concept, polynomial approximations for percentile functions using the Response Surface Methodology, and global optimization procedures based on the RLT in order to solve the resulting nonconvex polynomial programming approximating problem. This framework is generally applicable to engineering design problems involving nonconvex objective functions and constraints that are defined in terms of random variables and parameters following any continuous or discrete distributions. We have illustrated the proposed approach by applying this to the problem of reducing piston slap, an undesirable engine noise due to piston secondary motion.

As a further refinement of the proposed algorithm, note that the described procedure involves solving a single polynomial approximation problem constructed over the entire design space to global optimality. This approach can be further improved by employing a sequence of polynomial programming approximations to the problem coordinated via branch-and-bound techniques with specially designed node-partitioning schemes (see Sherali and Ganesan (2003) for a conceptually related methodology for solving black-box optimization problems). We did not pursue this for the present problem due to the limited amount of physical test data that was available, which precluded the derivation of Kriging approximations over partitioned subspaces in the design space. For example, in the case of the piston slap problem studied herein, the piston noise and friction measurement data obtained for the various design variable realizations involved time-consuming, expensive physical testing at a facility off-site to Ford Motor Company. The generation of additional data over subspaces in an interactive fashion would be prohibitive, time-and expense-wise, and would render this approach impractical. Nonetheless, given a sufficient cache of experimental data points, it would be worthwhile to automate this framework by constructing polynomial programming approximations over different hyperrectangles in the design space using the available data points, within the context of a partitioning approach. We propose investigating this, as well as other global optimization approaches, for solving such probabilistic design problems to (near-) global optimality for future research.

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